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EQUATIONS OF PLASMA PHYSICS

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Notes by Stephen Omohundro

(a) Plasmas

A plasma is a gas of charged particles under conditions where collective electromagnetic interactions dominate over interactions between individual particles. Plasmas have been called the fourth state of matter [1]. As one adds heat to a solid, it undergoes a phase transition (melting) to become a liquid. More heat causes the liquid to boil into a gas. Adding still more energy causes the gas to ionize (i.e. some of the negative electrons become dissociated from their gas atoms, leaving positively charged ions). Above 100,000°K, most matter ionizes into a plasma. While the earth is a relatively plasma-free bubble (aside from fluorescent lights, lightning discharges, and magnetic fusion energy experiments) 99.9% of the universe is in the plasma state (e.g. stars and most of interstellar space).

Though a plasma is a gas, the dominance of collective interactions makes the behavior quite different from that described by gas dynamics. The so-called plasma parameter $g = \left[\frac{8\pi e^2}{kT}\right]^{3/2}\sqrt{n}$ (e is the charge of an electron, n is the electron density and kT is the temperature in energy units) must be much less than unity for this "plasma approximation" to be valid. In this case, it is appropriate to replace an exact description of the position and velocity of every particle by a density $f_\alpha(x,v,t)\,dx\,dv$ in position-velocity phase space for every "species" of particle α (eg. electrons, different types of ions). This gives the number of particles of a given species per unit volume at a particular value of $(x,v) \in \mathbb{R}^6$. The exact description would make f_α a sum of δ-functions, one for each particle, but the collective nature of the plasma suggests that it is a useful idealization to assume the f_α's to be smooth. Notice that in hydrodynamics one usually assumes the velocity to be a function of position, whereas here we may have particles of many different velocities at each point in space.

Particles move in phase space for two reasons: they change their spatial position x if they have a non-zero velocity, and they change their velocity v if they feel a non-zero force. Here the only force we will consider is the electromagnetic Lorentz force $e_\alpha(E+\frac{v}{c}\times B)$ due to the electric field E and magnetic field B. These are vector fields on spatial $\mathbb{R}^3$. e_α is the charge of a particle of species α, and c is the speed of light (which appears because we are using Gaussian units) [2].

If the E and B fields were due to external agents, and the particles did not affect one another, then the evolution of the distribution f_α would be just transport along the flow in phase space generated by the Liouville equation:

$$\frac{\partial f_\alpha}{\partial t} + v\cdot\frac{\partial f_\alpha}{\partial x} + \frac{e_\alpha}{m_\alpha}\left[E + \frac{v}{c} \times B\right] \cdot \frac{\partial f_\alpha}{\partial v} = 0,$$

where m_α is the mass of a particle of species α, $\frac{\partial f_\alpha}{\partial x}$ is the gradient of f_α in the x directions, and $\frac{\partial f_\alpha}{\partial v}$ is the v-gradient. Here, E and B are prescribed vector-valued functions of space and time; given initial values for the f_α, one seeks their evolution in time.

The complexity of real plasma behavior is due to the fact that the E and B fields are affected by the plasma particles themselves. One thinks of the plasma as having a charge density

$$\rho_f(x) = \Sigma_\alpha e_\alpha \int f_\alpha(x,v)dv$$

and a current density

$$j_f(x) = \Sigma_\alpha e_\alpha \int f_\alpha(x,v)v dv.$$

These act as sources for the electromagnetic field, which evolves according to Maxwell's equations. The entire coupled set is called the Maxwell-Vlasov equations [3]:

$$\left.\frac{\partial f_\alpha}{\partial t} + v\cdot\frac{\partial f_\alpha}{\partial x} + \frac{e_\alpha}{m_\alpha}\left[E + \frac{v}{c} \times B\right] \cdot \frac{\partial f_\alpha}{\partial v} = 0 \quad\right\} \text{Vlasov}$$

$$\frac{1}{c}\frac{\partial E}{\partial t} = \nabla \times B - \left[\, \Sigma_\alpha\, e_\alpha \int f_\alpha(x,v)\, v\, dv \,\right] \frac{4\pi}{c}$$

$$\frac{1}{c}\frac{\partial B}{\partial t} = -\nabla \times E \qquad\qquad \text{Maxwell}$$

$$\nabla \cdot B = 0$$

$$\nabla \cdot E = 4\pi\, \Sigma_\alpha\, e_\alpha \int f_\alpha(x,v)\, dv$$

The electromagnetic field here is due to the smoothed particle distribution. If one wanted to include some particle discreteness effects, the zero in the Vlasov equation could be replaced by a term giving the change in f_α due to particle "collisions" (which are really just electromagnetic interactions that we ignored when we smoothed f_α). In this case the equation is known as the Boltzmann equation and the typical effect is to cause distributions to relax to "Maxwellian" form $f(x,v) \propto e^{-mv^2/2kT}$ at each point x.

(b) Simplifications

We may treat the Maxwell-Vlasov equations as an initial value problem where we are given initial conditions $f_\alpha(x,v)$, $E(x)$, and $B(x)$ consistent with the constraints, and we wish to solve for the time evolution. As yet, only a little is known rigorously about the existence, uniqueness and qualitative behavior of the solutions to these equations. To help with analysis, one looks at a series of simplified sets of equations which are expected to realistically model the physical behavior in restricted parameter regimes. Understanding the relations between the animals in this zoo of simplifications is itself a challenging mathematical problem.

I. If we are interested in slow electrostatic phenomena we may take the limit $c \longrightarrow \infty$. This eliminates magnetic effects and makes the electro-magnetic field a slave to the particles, with no independent evolution of its own. The resulting system of equations is known as the <u>Poisson-Vlasov</u> equations:

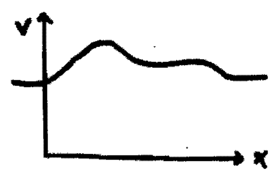
$$\frac{\partial f_\alpha}{\partial t} + v \cdot \frac{\partial f_\alpha}{\partial x} + \frac{e_\alpha E}{m_\alpha} \cdot \frac{\partial f_\alpha}{\partial v} = 0$$

$$\text{div } E = \rho^f, \qquad \text{curl } E = 0$$

Poisson's name is used here because curl $E = 0$ implies that E arises from a potential V satisfying Poisson's equation $\Delta V \propto \rho^f$. This same set of equations describes the evolution of a "gas" of massive bodies, like stars, under the influence of Newtonian gravitational forces. In this context something about the existence and uniqueness of solutions is known. In 4 spatial dimensions there is collapse for Newtonian interactions. In 2 dimensions the solution is fine for all time if the initial conditions are compactly supported [4]. Not much is known about the physically relevant 3 dimensional case.

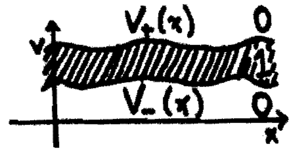
II. If we are dealing with a cold plasma, then there is not much dispersion in the velocity distribution function at each point. In this case it is reasonable to restrict attention to distributions that are δ-functions in the velocity direction:

$$f_\alpha(x,v,t) = \delta(v - v_\alpha(x,t))\, \rho_\alpha(x,t)$$



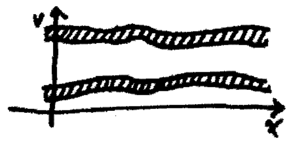
The evolution here is closely related to compressible fluid equations.

III. In both the stellar dynamics and the electrostatic plasma situations, people have studied so called "water-bag" models [5] where the distribution function is 1 in a region of x,v space and 0 outside it:

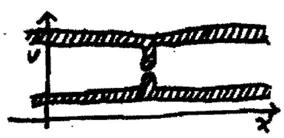


If we let this distribution evolve via the Liouville equations, it remains a water bag, but the bag might try to curl over itself:

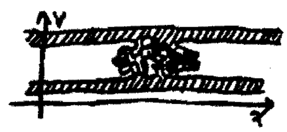
representing the formation of a shock wave. Water-bag models are good for numerical studies. For example, initial conditions representing 2 streams with different velocities



typically develop "fingers"

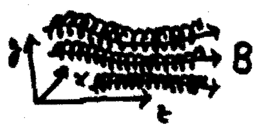


which sprout "tendrils" and get tangled up:

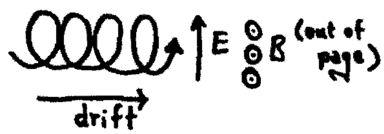


It is a challenge to understand this behavior, perhaps utilizing strange attractors.

IV. If the magnetic field is large and primarily external, then the particles like to wind around the field lines in tight little helices

(because of the v × B force): 

We can write the equations of motion for the so-called "guiding center" of these little loops. Considering a distribution function of guiding centers, we get a guiding center plasma. In the presence of external forces or gradients in the magnetic field, the guiding centers "drift" according to specifiable laws. For example, in the presence of an electric field, we get drifts in the direction of

E × B: 

If the equipotentials enclose bounded regions, then the drifts can flow

in loops: 

. Restricting to x,y and calling the guiding center density $\rho(x,y)$, the evolution in a constant magnetic field B_z and an electric field due to the particles and a uniform neutralizing background is given by $\rho_t = \langle \Delta^{-1}(\rho - 1), \rho \rangle$ where $\langle a,b \rangle = a_x b_y - a_y b_x$. This equation is precisely the same as that which describes vorticity evolution in 2 dimensional fluids. Much of the analysis of fairly artificial vorticity examples becomes quite realistic in this context. For example a vortex blob (vorticity 1 in a region, 0 outside it)  is very difficult

to approximate experimentally, but a guiding center blob is just a beam of particles. One finds a variety of complicated behaviors in the evolution of these blobs [6], including stationary patterns, fingers, and oscillations, similar to those seen for water bags. As another example, point vortices 

correspond to a collection of discrete beams.

V. The previous four simplifications were subsystems of the Maxwell-Vlasov system. The last simplification is a quotient instead. Instead of keeping track of the whole distribution function, $f_\alpha(x,v,t)$, we focus attention on the low order velocity moments, which turn out to be physically meaningful quantities. The zeroth moment is proportional to the charge density $\rho_\alpha(x,t) = e_\alpha \int f_\alpha(x,v,t)\, dv$ or mass density. The local average velocity is related to the first order moment by $\bar{v}_\alpha(x,t) = \dfrac{\int f_\alpha(x,v,t)\; v\; dv}{\int f_\alpha(x,v,t)\; dv}$. The tensor density p^α_{ij} is related to the second order moments by:

$$p^\alpha_{ij}(x,t) = \int (v_i - \bar{v}_i)(v_j - \bar{v}_j)\, f_\alpha(x,v,t)\, dv.$$

We have a map $f \longmapsto (\rho, v)$, and one would hope that the flow in f-space "covers" the flow in ρ,v-space. Unfortunately the (ρ,v) evolution equations contain p_{ij}, and the p_{ij} evolution depends on higher moments in a never-ending heirarchy. In practice, one truncates this hierarchy and makes the assumption that p_{ij} is a given function of ρ. Exactly this assumption is made in obtaining the equations for compressible fluids, and so the reduced equations are called the 2-fluid model for the plasma.

It is a general problem to understand the approximation processes here, perhaps in the way that the relationship between compressible and incompressible fluid descriptions was elucidated in Majda's talk in this seminar. An understanding of how certain properties are preserved has come from a geometric picture that has developed in the last 2-3 years, based on one which Arnold painted for inviscid fluids about 15 years ago [7]. We describe this development in the following section.

(c) Hamiltonian Structure

Poisson manifolds are a generalization of symplectic manifolds, which have been used in geometric mechanics for about 30 years. A Poisson manifold is a manifold M with a Poisson bracket $\{,\}$ defined so as to make $C^{\infty}(M)$ a Lie algebra and additionally to satisfy a Leibniz rule: $\{fg,h\} = f\{g,h\} + g\{f,h\}$. More geometrically, this gives rise to a bilinear form B on the cotangent bundle, defined by $B(df,dg) = \{f,g\}$. (Many aspects of this structure were discovered by Lie [8] and were recently rediscovered by various people [9] with no reference to Lie.) B gives rise to a map $\tilde{B}: T^{*}M \longrightarrow TM$. If B is non-degenerate, we may invert this map to get a symplectic 2-form. Poisson structures are more interesting because they <u>can</u> be degenerate.

The range of $\tilde{B}$ is a linear subspace of each fibre in TM. Thus it is like a distribution, but the dimension jumps as we move around in M. The Jacobi identity for $\{,\}$ implies that the Frobenius integrability condition is satisfied, thus, in so-called <u>regular</u> regions where the range of $\tilde{B}$ is of constant dimension, M is foliated by smooth leaves. Since $\{,\}$ restricted to these leaves is non-degenerate, they have a natural symplectic structure. In fact, the symplectic leaves exist even at non-regular points, so Poisson manifolds are a promising model for the Hamiltonian formulation of physical situations with parameters, in which the number of variables changes at certain limiting values (e.g. $c \longrightarrow \infty$).

Poisson manifolds are the home of Hamiltonian systems. If $H: M \longrightarrow \mathbb{R}$ is a Hamiltonian function, then we call $\tilde{B}(dH) \equiv \xi_H$ the Hamiltonian vector field associated to it. For an arbitrary function F we have $\xi_H \cdot F = \{F,H\}$ so $\dot{F} = \{F,H\}$ defines the flow of ξ_H. In coordinates x_i this says

$$\dot{x}_i = \{x_i,H\} = \sum_{j=1}^{\dim M} \{x_i,x_j\} \frac{\partial H}{\partial x_j} .$$

This form of Hamilton's equations is due to Poisson. Note that all Hamiltonian vector fields are tangent to symplectic leaves.

In this framework, we may formulate the stability criterion of Lagrange and Arnold [7]. If $p \in M$ is an equilibrium point for ξ_H, then dH restricted to the symplectic leaf containing p must vanish at p. If d^2H restricted to this leaf is definite, then p is a stable equilibrium point for the flow restricted to the leaf. If in addition p is in the regular region, then stability holds for the entire flow.

There is a natural Poisson bracket called the Lie-Poisson bracket defined on $\mathfrak{g}^*$, the dual of a Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a group G, then we may identify left invariant functions on T^*G with functions on $\mathfrak{g}^*$. T^*G has the natural canonical Poisson bracket which is preserved by left translations, so the space of left invariant functions is closed under the operation of Poisson brackets. Identifying this subspace with $C^\infty(\mathfrak{g}^*)$, we obtain the Lie-Poisson bracket:

$$\{F,G\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right] \right\rangle$$

where $\mu \in \mathfrak{g}^*$, $F,G \in C^\infty(\mathfrak{g}^*)$, $\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \in \mathfrak{g}$ and $\langle\ ,\ \rangle$ is the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. In terms of the structure constants c_{ijk} for $\mathfrak{g}$ this is:

$$\{F,G\}(\mu) = \sum_{ijk} c_{ijk}\, \mu_i \frac{\partial F}{\partial \mu_j} \frac{\partial G}{\partial \mu_k} .$$

The symplectic leaves in $\mathfrak{g}^*$ of this Poisson bracket are exactly the orbits of the coadjoint representation of G in $\mathfrak{g}^*$. This fact was "discovered" by Kirillov and others ~ 1960, but Lie had the result in 1890. Lie also showed that any homogeneous symplectic manifold is one of these leaves, thus "prediscovering" further results of the 1960's.

Poisson bracket preserving maps $J: M \longrightarrow \mathfrak{g}^*$ from a Poisson manifold to the dual of a Lie algebra are called momentum mappings. By evaluation, such a map gives rise to a map $\mathfrak{g} \longrightarrow C^\infty(M)$. Viewing the image of an element of $\mathfrak{g}$ as a Hamiltonian function, such maps are associated with Poisson bracket preserving actions of G on M. Usually one goes the other way. Start with a G action on M, get the associated momentum mapping, and then notice that it is G equivariant. If M is a homogeneous symplectic manifold, then the momentum mapping must be a covering onto a coadjoint orbit.

All of the plasma equations mentioned above have Poisson structures. A structure for the Maxwell-Vlasov equations was found by Morrison [10]. An error [11] was corrected by Marsden and Weinstein [12].

For the Poisson-Vlasov equations, we consider the infinite-dimensional Lie algebra $\mathfrak{g} = C^\infty(\mathbb{R}^6)$ of functions on $\mathbb{R}^6 = (x,v)$ with the canonical Poisson bracket:

$$\{a,b\} = \Sigma_i\left[\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial v_i} - \frac{\partial a}{\partial v_i}\frac{\partial b}{\partial x_i}\right].$$

The associated group is the group of diffeomorphisms of $\mathbb{R}^6$ that preserve $\{,\}$. The dual space $\mathfrak{g}^* = \mathcal{D}'(\mathbb{R})$ is the space of distributions on $\mathbb{R}^6$ which we identify with the space of plasma distributions. Using the Lie-Poisson bracket on $\mathfrak{g}^*$, the flow generated by the Hamiltonian

$$H = \frac{1}{2}\int mv^2 f(x,v)\, dx\, dv + \frac{1}{2}\int E^2\, dx$$

is exactly the Poisson-Vlasov evolution.

The subspace h of g given by affine linear functions of the velocity $\Sigma_i\ a_i(x)v_i + b(x)$ is actually a subalgebra. Working out the Poisson bracket:

$$[\ \Sigma_i\ a^1_i(x)v_i + b^1(x),\ \Sigma_i\ a^2_i(x)v_i + b^2(x)]$$

$$= \Sigma_i \left(\Sigma_j \left[a^2_j \frac{\partial a^1_i}{\partial x_j} - a^1_j \frac{\partial a^2_i}{\partial x_j}\right]\right) v_i + \Sigma_j \left(a^2_j \frac{\partial b^1}{\partial x_j} - a^1_j \frac{\partial b^2}{\partial x_j}\right)$$

we see that this subalgebra is isomorphic to the semi-direct product $\mathfrak{X}(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3)$, where $\mathfrak{X}(\mathbb{R}^3)$ is the Lie algebra of vector fields on $\mathbb{R}^3$, acting by differentiation on $C^\infty(\mathbb{R}^3)$. This Lie algebra is associated with the semidirect product $\mathrm{Diff}(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3)$ of the group of diffeomorphisms of $\mathbb{R}^3$ acting by pullback on the additive group of functions on $\mathbb{R}^3$. This is the group used in the description of compressible fluids [13]. It has also appeared in the "current algebra" approach to quantum field theory [14].

The injection $h \longrightarrow g$ gives us a natural surjection $g^* \longrightarrow h^*$, which is a momentum mapping for the action of the fluid group on the Vlasov plasma manifold. While we have the relation between the Vlasov and fluid Poisson manifolds, complete understanding of the dynamics requires understanding of how the Hamiltonians relate.

Some of the simplifications which were subsystems turn out to be motions on coadjoint orbits. For example the water bag states with fixed area form a coadjoint orbit in the one-dimensional case. The Poisson bracket is just the one usually associated with the KdV equation [15], but the Hamiltonian is different.

(d) Stability

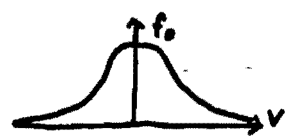
Another set of interesting mathematical questions may be asked regarding the stability of equilibrium configurations. A good reference for the physics literature earlier than 1960 is [16]. The most detailed work has been done on the 1-dimensional Poisson-Vlasov equations:

$$\frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} + E\,\frac{\partial f}{\partial v} = 0,$$

$$\frac{\partial E}{\partial x} = \int f(x,v)\,dv - 1$$

where the 1 represents an immobile neutralizing background. The distributions $f_0(x,v) = f_0(v)$, $E = 0$, $\int f_0(v)\,dv = 1$ are equilibrium solutions. For the stability analysis we linearize around this fixed point. With Landau [17] we Fourier analyze in x and Laplace transform in t. (This was the beginning of Landau's work with moving poles in particle physics.) One finds that for smooth f the perturbed E-field damps out, an effect now called "Landau damping".

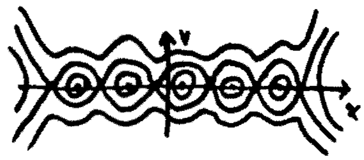
In 1955 N.G. Van Kampen [18] found solutions which are traveling waves for all time. These solutions, however, are not smooth and have the form $g(x,v) = e^{ikx}\left[a\,P\frac{1}{v-V} + b\,\delta(v-V)\right]$. One can connect the two pictures by integrating over the Van Kampen modes. One finds that if the initial f_0 decreases with $|v|$,



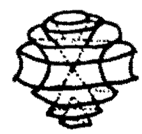
then the solutions to the linear equations remain bounded. If there are bumps in the distribution (beams) then one may find instabilities. In 1960 Penrose [19] found a criterion for stability using the argument principle in locating roots: there is instability when f_0 has a local minimum V such that

$$\int \frac{f(v) - f(V)}{(v-V)^2}\,dv > 0.$$

It is a delicate matter to go from neutral linear stability to stability of solutions of the full nonlinear equations. In 1958 Berstein-Greene-Kruskal [20] discovered so called BGK modes. The density of the plasma is constant on level surfaces of the Hamiltonian.

These level curves may look like 

in a reference frame moving with the wave. The density variations exactly produce the electric field necessary to support themselves. These modes are smooth, but as the amplitude gets smaller they may approach the (singular) Van Kampen modes.

It is interesting to try to apply Arnold's stability criterion in this situation. His condition only applies at a regular point of the Poisson structure, since if the dimension of the symplectic leaves goes up when we perturb, there are more directions that we must check for instability. For example, the symplectic leaves for the Lie-Poisson bracket on $sl(2)^*$ look like:  Perturbing from the origin puts us on a 2 dimensional leaf where the dynamics could take us far away. As in the regular point case, we need make no requirements on the Hamiltonian in directions transverse to the symplectic leaves in which the dimension doesn't change.

In the plasma case at hand, the symplectic leaf has infinite codimension. It turns out that we can get by with restricting to only a codimension 2 surface, defined by level sets of two functions. Anything of the form $\int G(f(x,v))\, dx\, dv$ is constant on coadjoint orbits. For the first function we may choose the total number of particles $\int f\, dx\, dv$. With this choice, the 2nd function is determined by the equilibrium point we are interested in. For the case of a Maxwellian distribution αe^{-v^2} the function turns out to be the entropy: $\int f \log f\, dx\, dv$. This is related to the fact that the distribution function which maximizes the entropy for given total particle number is the Gaussian. This line of thinking has only begun, and there is lots left to be done.

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