A Hamiltonian Approach to Perturbation Theory

BY
STEPHEN
OMOHUNDRO

presented at:

Workshop on Methods of Nonlinear Dynamics

Naval Surface Weapons Laboratory

July 23-26, 1984
Washington, D. C.

This work was supported by the Office of Basic Energy Sciences of the U. S. Department of Energy under Contract No. DE-AC03-76SF00098.

OUTLINE

1. INTRODUCTION

2. NON-SINGULAR PERTURBATION THEORY

3. GEOMETRIC HAMILTONIAN MECHANICS

4. HAMILTONIAN PERTURBATION THEORY

5. SINGULAR PERTURBATION THEORY: mechanical systems

6. SINGULAR PERTURBATION THEORY: wave systems

1808: Lagrange introduces Hamilton's equations to simplify perturbation analysis.

Late 1800's: Hamiltonian Mechanics is refined and the connection with optics and variational principles is made.

~1900: Poincaré develops Hamiltonian perturbation methods using generating functions and the geometric approach to dynamics.

1918: Emmy Noether makes connection between symmetries and conserved quantities.

1920's: The development of quantum mechanics rests heavily on the Hamiltonian framework.

1960's: Development of coordinate free Hamiltonian mechanics using symplectic geometry and Lie-transform perturbation theory.

1970's: Using geometric methods, Hamiltonian structures are found in virtually all areas of physics.

Some Systems with a Hamiltonian Formulation

Quantum Mechanics
(both Heisenberg and Schrödinger pictures)

Fluid Mechanics
(both compressible and incompressible)

Maxwell's Equations

Elasticity Theory

Louiville equation

Vlasov's Equations for Plasmas
(both Maxwell-Vlasov and Poisson-Vlasov)

Magneto-hydrodynamics

Multi-fluid plasmas

Superfluids and Superconductors

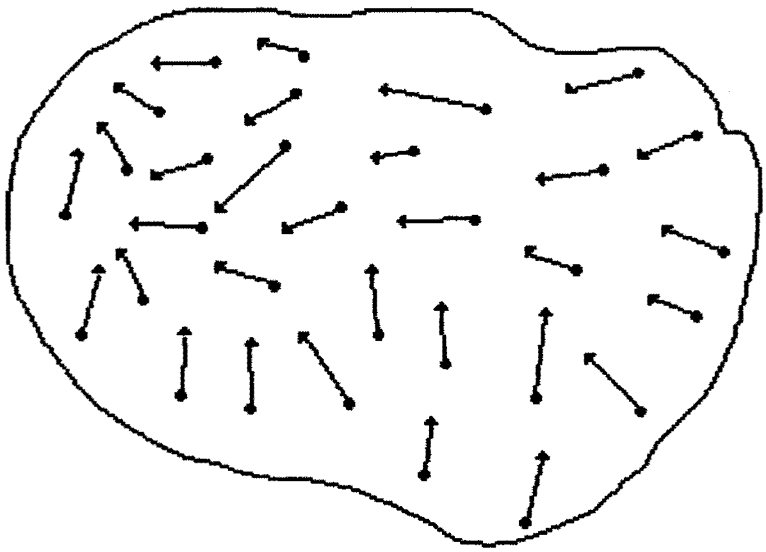
General Relativity

Chromohydrodynamics

Korteweg de Vries

etc. etc. etc.

DYNAMICAL SYSTEM



STATE SPACE = MANIFOLD

DYNAMICS = VECTOR FIELD

PERTURBATION THEORY

unperturbed:

$$\dot{x} = X_0 \qquad x(t=0) = y_0$$

solution flow: $x(t, y_0)$

perturbed system:

$$\dot{x} = X(\varepsilon) = X_0 + \varepsilon X_1 + \dots.$$

$$x(t=0) = y(\varepsilon) = y_0 + \varepsilon y_1 + \dots.$$

solution flow: $x(t, \varepsilon, y_0)$

If $X(\varepsilon) = X_0 + \varepsilon X_1 + \dots.$ is a unifor asymptotic expansion then for fixed or bounded t, $x(t, \varepsilon, y_0)$ has a uniform expansion :

$$x(t, \varepsilon, y_0) = x_0(t, \varepsilon, y_0) + \varepsilon x_1(t, \varepsilon, y_0) + \dots.$$

where:

$$\dot{x}_0(t, \varepsilon, y_0) + \varepsilon \dot{x}_1(t, \varepsilon, y_0) + \dots.$$

$$= X_0(t, x_0 + \varepsilon x_1) + \varepsilon X_1(t, x_0 + \varepsilon x_1) + \dots.$$

This holds term by term in ε.

$$\dot{x}_0^a + \varepsilon \dot{x}_1^a + \frac{\varepsilon^2}{2!}\dot{x}_2^a + \ldots =$$

$$X_0^a(\dot{x}_0 + \varepsilon \dot{x}_1 + \frac{\varepsilon^2}{2!}\dot{x}_2 + \ldots) +$$

$$\varepsilon X_1^a(\dot{x}_0 + \varepsilon \dot{x}_1 + \frac{\varepsilon^2}{2!}\dot{x}_2 + \ldots) +$$

$$\frac{\varepsilon^2}{2!} X_2^a(\dot{x}_0 + \varepsilon \dot{x}_1 + \frac{\varepsilon^2}{2!}\dot{x}_2 + \ldots) + \ldots$$

PERTURBATION EQUATIONS:

$$\dot{x}_0^a = X_0^a(x_0)$$

$$\dot{x}_1^a = \sum_{b=1}^{2N} \frac{\partial X_0^a}{\partial x^b}(x_0)\cdot x_1^b + X_1^a(x_0)$$

$$\dot{x}_2^a = \sum_{b,c=1}^{N} \frac{\partial^2 X_0^a}{\partial x^b \partial x^c}(x_0)\, x_1^b\, x_1^c + \sum_{b=1}^{N} \frac{\partial X_0^a}{\partial x^b}(x_0)\cdot x_2^b$$

$$+ 2\sum_{b=1}^{N} \frac{\partial X_1^a}{\partial x^b}(x_0)\cdot x_1^b + X_2^a(x_0)$$

$$\vdots$$

How do we interpret these perturbation equations

GEOMETRICALLY ?

The first order system is:

$$\dot{x}_0^a = X_0^a(x_0)$$

$$\dot{x}_1^a = \sum_{b=1}^{2N} \frac{\partial X_0^a}{\partial x^b}(x_0)\cdot x_1^b + X_1^a(x_0)$$

$$x_0(t=0)=y_0 \qquad x_1(t=0)=y_1$$

Geometrically, x_0 coordinatizes the manifold M and x_1 coordinatizes the fiber over x_0 in the tangent bundle TM.

x_0 lifts to TM as:

$$\tilde{X}_0 = \left.\frac{d}{dt}\right|_{t=0} Tx_0(t)$$

x_1 lifts as:

$$\tilde{X}_1(x,v) = \left.\frac{d}{dt}\right|_{t=0} \left(v + t\, X_1(x)\right)$$

the perturbation dynamics on TM is:

$$\tilde{X}_0 + \tilde{X}_1$$

$x(\epsilon)$ is an ϵ dependent point, i.e. a path parameterized by ϵ in M.

Introduce: $P_1 M$ = paths of the form:

$$p: I \to I \times M \qquad p: \epsilon \mapsto (\epsilon, x(\epsilon))$$

Then define: $P_0 M$ = equivalence classes:

$$p_1 \sim p_2 \quad \text{iff} \quad p_1(0) = p_2(0)$$

Then define: $P_a M$ = equivalence classes:

$$p_1 \sim p_2 \quad \text{iff} \quad p_1(\epsilon) = p_2(\epsilon) \quad 0 \le \epsilon \le a$$

$$P_1 M \to P_a M \to P_b M \to P_0 M \quad \text{if} \quad 1 \ge a \ge b \ge 0$$

Then define: GM = equivalence classes:

$$p_1 \sim p_2 \text{ iff } \exists a \text{ s.t. } p_1(\epsilon) = p_2(\epsilon) \quad \forall \quad 0 \le \epsilon \le a$$

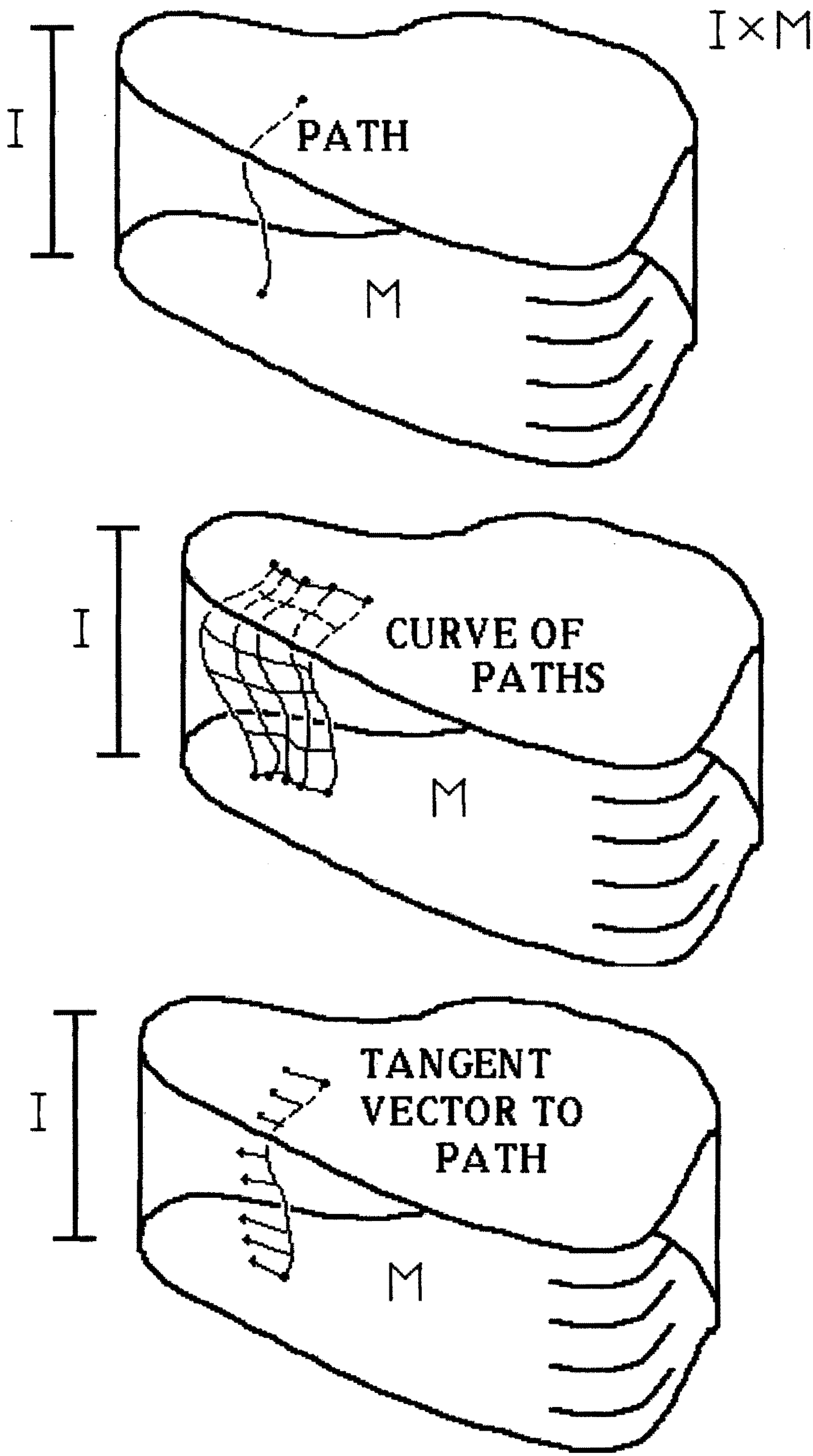
$$P_a M \to GM \qquad P_0 M \text{ if } \quad a > 0$$

Finally define: JM = equivalence classes:

$p_1 \sim p_2$ iff $\forall$ C^∞ functions f on $I \times M$ we have:

$$\frac{\partial^i}{\partial \epsilon^i} f(p_1(\epsilon)) = \frac{\partial^i}{\partial \epsilon^i} f(p_2(\epsilon)) \qquad 0 \le i \le J$$

$$GM \to \infty M \to IM \to JM \to P M \quad \text{if} \quad I > J$$


I×M
I
PATH
M
I
CURVE OF PATHS
M
I
TANGENT VECTOR TO PATH
M

Tangent vectors on jet space are defined by the natural projection from path space.

Tangent vectors with the first J derivatives at $\epsilon = 0$ vanishing are annihilated.

Thus the tangent space to JM is a quotient of one on P_1M.

Coordinatize:

M by x^a $\qquad$ P_1M by $x^a(\epsilon)$

JM by $x_0^a, x_1^a, \ldots, x_J^a$ representing the class of the curve: $x_0^a + \epsilon x_1^a + \ldots + \frac{\epsilon^J}{J!} x_J^a$

Then coordinatize TJM by $\{x_0^a, \ldots, x_J^a, v_0^a, \ldots, v_J^a\}$ representing $\sum_{a=1}^{N} v_0^a \frac{\partial}{\partial x_0^a} + \ldots + v_J^a \frac{\partial}{\partial x_J^a}$

The coordinates of the class of $V^a(\epsilon)$ are then:

$$v_k^a = \left.\frac{\partial^k}{\partial \epsilon^k}\right|_{\epsilon=0} V^a(\epsilon)$$

Dynamics: $\dot{x} = X(\varepsilon, x)$ on M

lifts to path space as $\tilde{X}$ where:

$$\tilde{X}(p) \text{ is } X(\varepsilon, p(\varepsilon))$$

This leaves all equivalence classes intact and so projects down to dynamics on the quotient spaces. On JM:

$$V_k^a(x_0, \ldots, x_J) = \left.\frac{\partial^k}{\partial \varepsilon^k}\right|_{\varepsilon=0} X^a(\varepsilon, x_0 + \varepsilon x_1 + \ldots + \frac{\varepsilon^J}{J!} x_J)$$

Which is exactly the perturbation dynamics to order J.

Traditional Hamiltonian mechanics utilizes:

generalized coordinates q_i

and their

conjugate momenta p_i .

The Poisson bracket of two functions is defined as:

$$\{f,g\}=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

The Hamiltonian is a function:

$$H(q_1,\ldots,q_n,p_1,\ldots,p_n)$$

Any Observable f evolves via:

$$\dot{f}=\{f,H\}$$

A Poisson Manifold is:

a manifold

with a
Poisson Bracket
on it.

A **Poisson Bracket** is a bilinear map from pairs of functions to functions which makes the space of functions a **Lie Algebra:**

1. Bilinear: $\{af+bg, h\}$
$=a\{f, h\} + b\{g, h\}$

2. Antisymmetric:

$$\{f, g\} = -\{g, f\}$$

3. Jacobi's Identity:

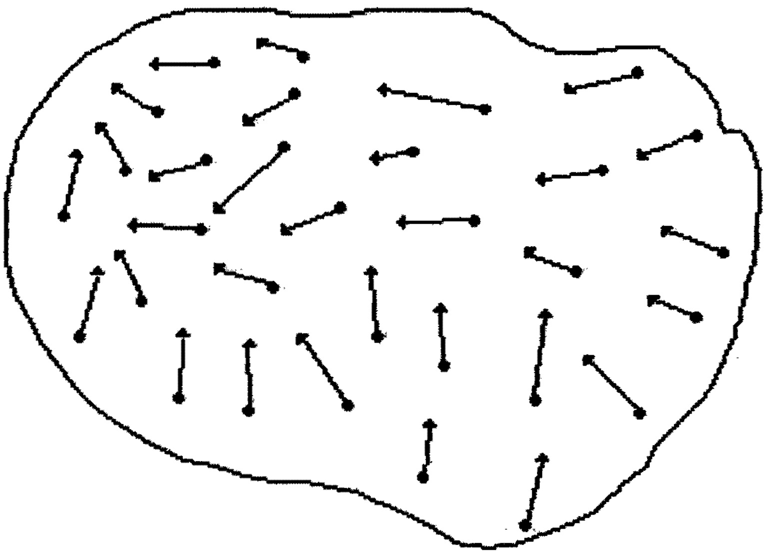
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

and in addition, acts like a derivative:

4. Derivation:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g \quad .$$

The Hamiltonian is a function on the Poisson manifold.



Dynamics is obtained from the Hamiltonian H and the Poisson bracket via:

$$\dot{z}^i = X_H \cdot z^i = \{z^i, H\}$$

X_H is the Hamiltonian vector field associated with H.

With coordinates z_i, the derivation property implies:

$$\{f, g\} = \sum_{i,j} \frac{\partial f}{\partial z_i} \{z_i, z_j\} \frac{\partial g}{\partial z_j}$$

The Poisson bracket is equivalent to an antisymmetric contravariant two-tensor:

$$J_{ij} \equiv \{z_i, z_j\}$$

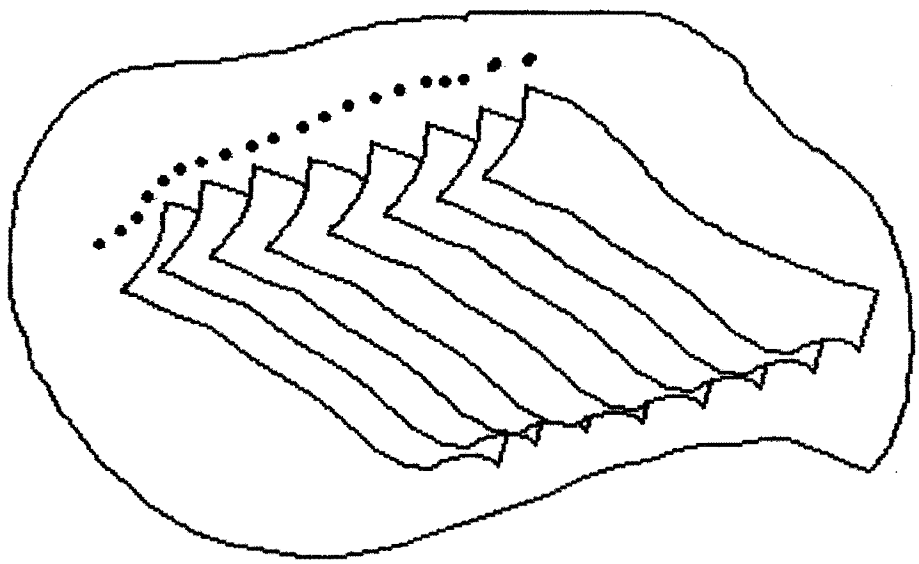
If this is nondegenerate, its inverse: $\omega \equiv J^{-1}$ is a closed, nondegenerate two-form called a symplectic structure.

In this case, we have a

Symplectic Manifold.

If J is degenerate, there are directions in which no Hamiltonian vector field can point.

Dynamics is restricted to the Symplectic leaves and Symplectic bones which stratify any symplectic manifold

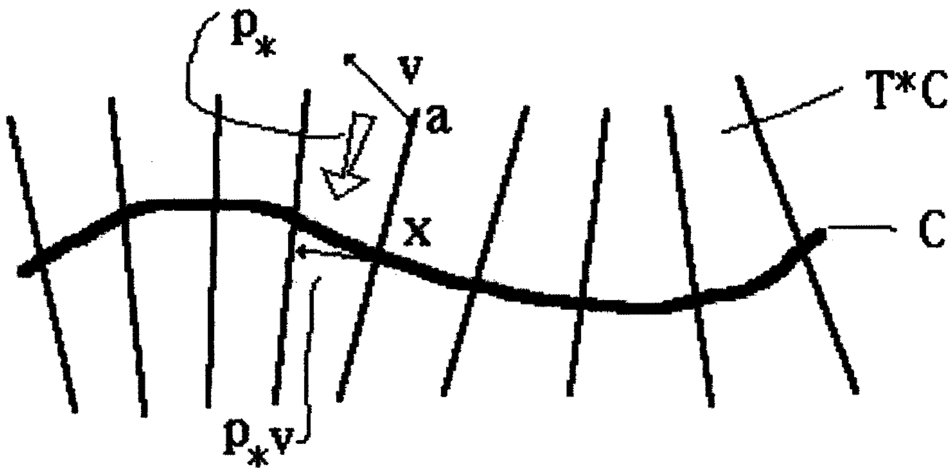


Functions which Poisson commute with everything are called:

Casimir functions

A natural symplectic manifold arises from Lagrangian systems on a configuration space C. L lives on TC. Hamiltonian mechanics takes place on the cotangent bundle: T*C.

This has a natural symplectic structure w. First we define the canonical one form t.



The one form t is to act on tangent vectors v to T*C at points (x,a) where a is a one-form on C. The natural projection p from T*C to C has a differential p_* which pushes v down to p_*v on C. The one form a can then eat this to give the desired value.

Thus: $t(v)=a(p_* v)$

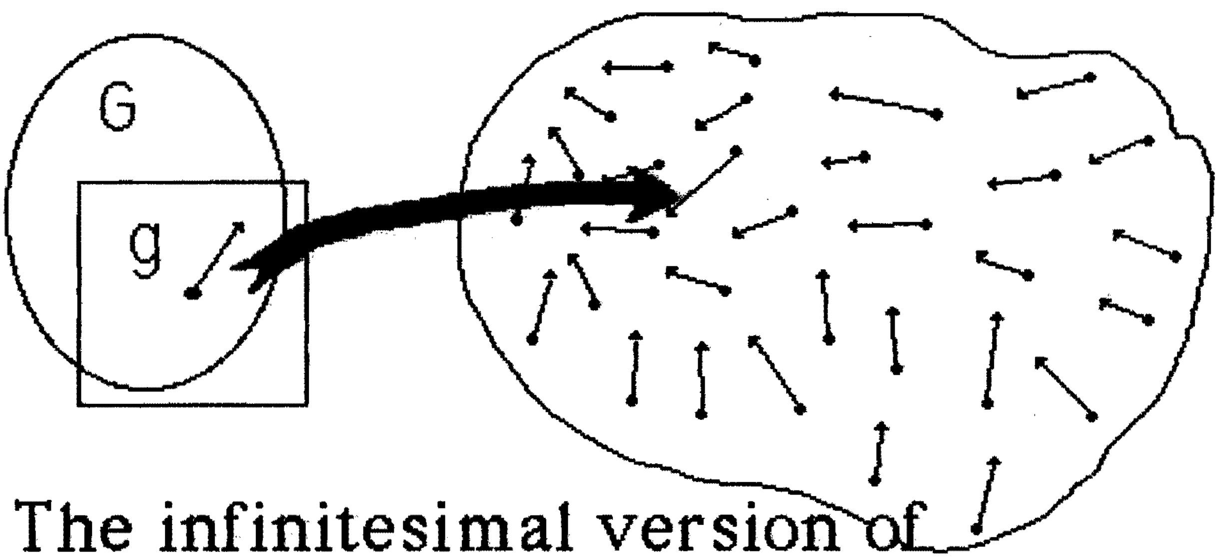
The canonical two form is then $w=-dt$.

This generalizes the usual canonical coordinates q,p to manifolds.

A Hamiltonian system with Symmetry consists of:

- A Poisson manifold M
- A Hamiltonian H
- A group G

where the group acts on M so as to preserve both {,} and H.



The infinitesimal version of this action is a homomorphism from G's Lie algebra to the Lie algebra of Hamiltonian vector fields on M.

Because these vector fields are Hamiltonian, we get a Lie algebra homomorphism from $\mathfrak{g}$ to functions on M under $\{,\}$.

At each point of M this associates a number with each element of $\mathfrak{g}$ in a linear fashion.

Thus we get a map from M to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$.

This map J is called the

Momentum map

for this group action.

The momentum map for translations is linear momentum and for rotations is angular momentum, for mechanical systems.

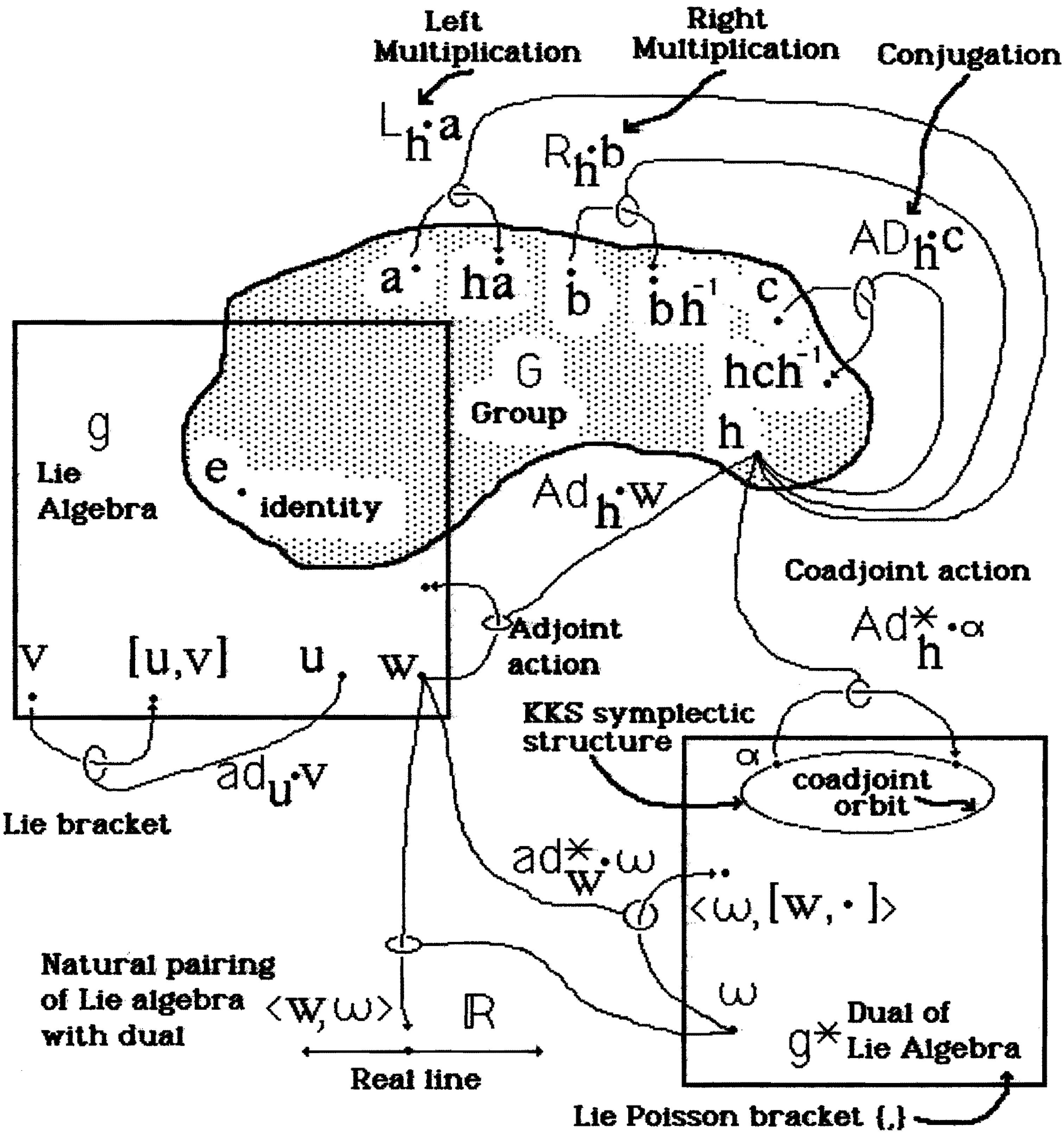
Noether's Theorem says that the value of the momentum map for a Hamiltonian symmetry is a constant of the motion.

For a one-dimensional symmetry this is shown by:

$$\dot{J} = \{J, H\} = -\{H, J\} = X_J H = 0$$

For larger groups, just choose a basis and use the above on each element.

Some Natural Group Actions:



We require that the momentum map be equivariant with respect to the coadjoint action:

$$J(g \cdot x) = Ad_g^* \cdot J(x)$$

eg. Angular momenta transform like vectors under rotation.

There are two ways of simplifying the dynamics when there is symmetry:

1. Restrict attention to J = constant

2. Project the dynamics down to the orbit space of the group action.

For noncommutative groups these two cannot be done together in general:

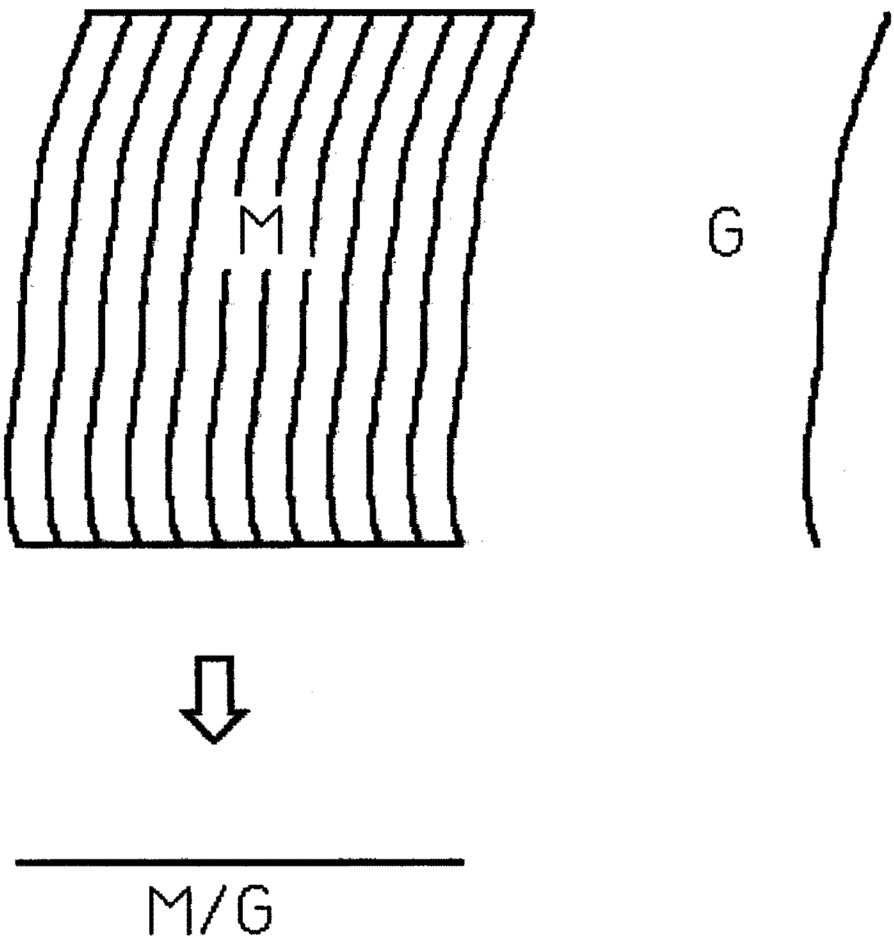
1. **Only the isotropy subgroup of J under the coadjoint action acts on J=const., but we may consider the orbit space of this.**

2. **J is not constant on orbits, and so we cannot restrict to J=const. on orbit space. But we can restrict to J inverse of a particular coadjoint orbit.**

The orbit space M/G has a natural Poisson structure.

Two functions on M/G pull back to G invariant functions on M. Their Poisson bracket is also invariant and so pushes down to M/G.

If H is G-invariant, then the dynamics is, and so pushes down to M/G.



1. The symplectic leaves and bones of M/G project to the coadjoint orbits of g^* under the momentum map.
   If α is an element of g^* and O_α is the coadjoint orbit through a, then $J^{-1}(O_\alpha)$ is a symplectic manifold called the reduced space at α. The dynamics projects to it and is Hamiltonian w.r.t. the restriction of H.

2. We obtain the same space by considering the orbit space of the isotropy subgroup G_α and restricting to $J=\alpha$.

   For commutative groups we elimiate two dimensions for each dimension of symmetry, for non-commutative groups, less.

An important general example of reduction is systems whose phase space is the cotangent bundle of a group, and whose Hamiltonian is left (or right) invariant under the canonical lift of group multiplication.

The orbits have one point in each fiber and so we may identify the orbit space with the cotangent space at the identity, i.e. the dual of the Lie algebra. The momentum map is then the identity and so the coadjoint orbits receive a natural symplectic structure called the Kirillov-Kostant-Souriou structure.

They are the symplectic leaves and bones of the Lie-Poisson bracket on the dual of the Lie algebra defined as:

$$\{f,g\}(a)=\langle a,[\frac{\delta f}{\delta a},\frac{\delta g}{\delta a}]\rangle$$

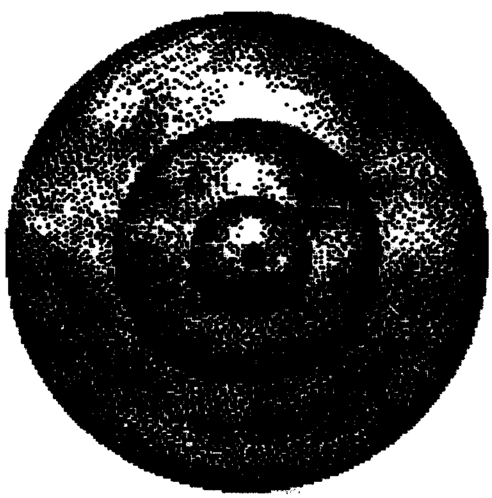
This structure is behind most of the recently discovered brackets.

Example: Rigid Body

The phase space is the cotangent bundle of the rotation group (three Euler angles and three angular momenta).

The Hamiltonian is rotationally symmetric.

Reduction leads to dynamics in the dual of the Lie algebra of the rotation group, representing the angular momenta in a body fixed frame. These are Euler's equations. The coadjoint orbits are spheres and the magnitude of the angular momentum is a Casimir function



Example: Perfect Fluid

The phase space is the cotangent bundle of the group of diffeomorphisms of the fluid region to itself (the configuration of the fluid points and the momentum density).

The Hamiltonian is invariant under interchange of fluid elements.

Reduction leads to dynamics in the dual of the Lie algebra: Eulerian momentum density.

The dynamics there is Euler's equations.

Example: Plasmas

Group is symplectomorphisms of particle phase space. Lie algebra is Hamiltonian vector fields, or equivalently, functions with Poisson bracket. Dual is densities on phase space, representing Vlasov particle densities.

One coadjoint orbit is orbit of delta function. Shows any symplectic manifold is a coadjoint orbit in dual of Lie alg.

Delta function on loop shows space of all loops of given action is a symplectic manifold.

There is a natural symplectic structure on the path space PM:

$$\tilde{w}_p(\tilde{V}_1, \tilde{V}_2) = \int_0^1 w_{p(\varepsilon)}(V_1(\varepsilon, p(\varepsilon)), V_2(\varepsilon, p(\varepsilon)))d\varepsilon$$

This gives the path dynamics with the Hamiltonian:

$$\tilde{H}(p) = \int_0^1 H(\varepsilon, p(\varepsilon))\, d\varepsilon$$

So the path space dynamics is Hamiltonian.

The perturbation dynamics on JM is also Hamiltonian.

If the bracket on M is $\{x^a, x^b\} = J^{ab}$ then the bracket on JM is:

$$\{x^a_k, x^b_m\} = J^{ab} \frac{k!m!}{J!} \delta_{k,J-m}$$

and the Hamiltonian:

$$\overline{H}(x_0, \ldots, x_J) = \frac{d^J}{d\epsilon^J}\bigg|_{\epsilon=0} H(\epsilon, x_0 + \epsilon x_1 + \ldots + \frac{\epsilon}{J!} x_J)$$

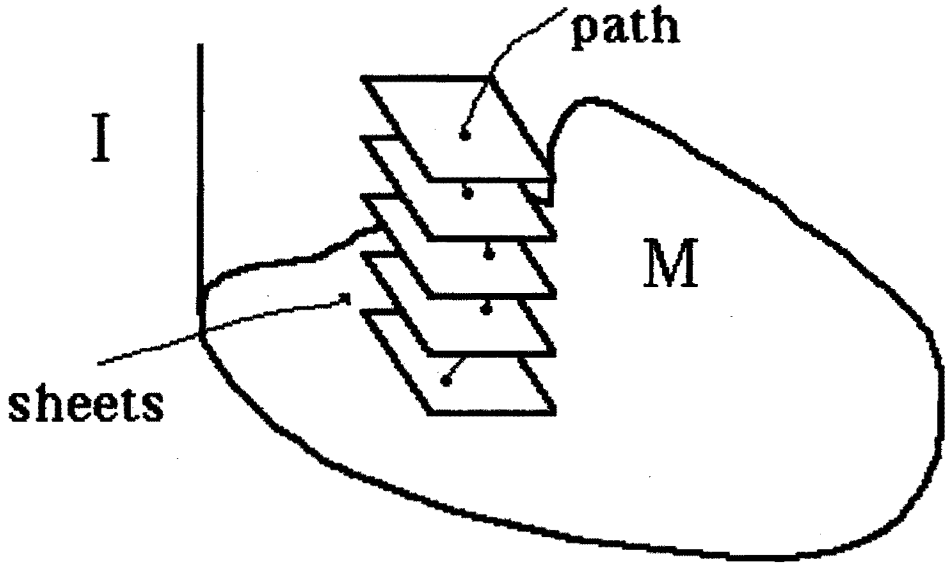
Together these give the correct perturbation dynamics.

ITERATED TANGENT BUNDLES

The tangent bundle to a symplectic manifold has a natural symplectic structure. If w is the structure on M then use it to identify TM and T*M. The desired structure is then the canonical one on T*M pulled back to TM.

This operation may be iterated to give symplectic structures on the iterated tangent bundles: TTM, TTTM, TTTTM,....

The Jth order jets naturally imbed in the Jth iterated tangent bundle. If the symplectic structure is pulled back to the jet space, one obtains the jet Poisson bracket given earlier.



Choose J sheets spaced evenly in I x M. The path dynamics projects down to the product of these sheets. We may map this structure to JM with arbitrary coefficients. If these coefficients are chosen to give a nonsingular result as the sheet spacing goes to zero, we again obtain the jet symplectic structure and Hamiltonian.

This shows that the perturbation bracket and Hamiltonian are in essence Jth derivatives of the path structures.

A Hamiltonian G action on M lifts to both the path space PM and the jet space JM and the corresponding momentum maps are equivariant.

We may introduce the group of paths in G and the group of J-jets of paths in G. These act naturally on PM and JM.

If M is a coadjoint orbit in the dual of G's Lie algebra then PM and JM are coadjoint orbits in the dual of the Lie algebras of the path group and the jet group. The corresponding KKS symplectic structures are our path and jet structures.

Reduction works as well. The reduced path space is the path space of the reduced space and the reduced jet space is the jet space of the reduced space.

I believe that the method of Lie transforms has a natural formulation in this context.

The Method of Averaging

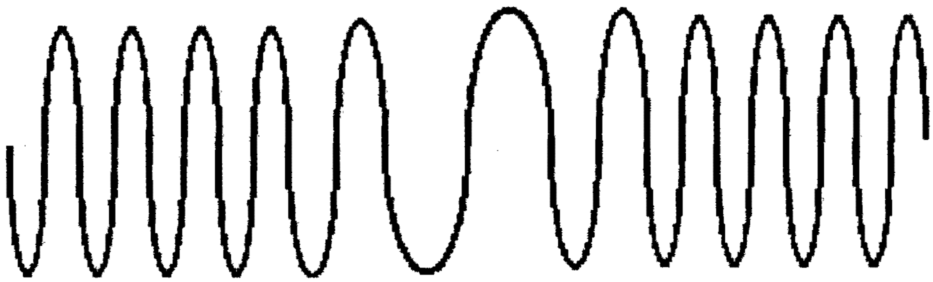
If the unperturbed system has all orbits periodic, then we can get asymptotic solutions good for times $t = \varepsilon^{-1}$.

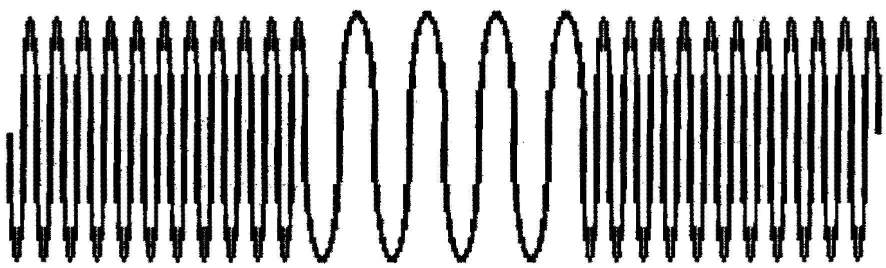
One pushes forward the perturbed vector field by the unperturbed flow and averages.

If the system is Hamiltonian, then the unperturbed system is symmetric under a circle action. The perturbation breaks the symmetry preventing reduction. The averaged system averages the first order perturbation Hamiltonian around the unperturbed loops to obtain the averaged dynamics. This is symmetric. Reduction gives the loop dynamics as a Hamiltonian system.

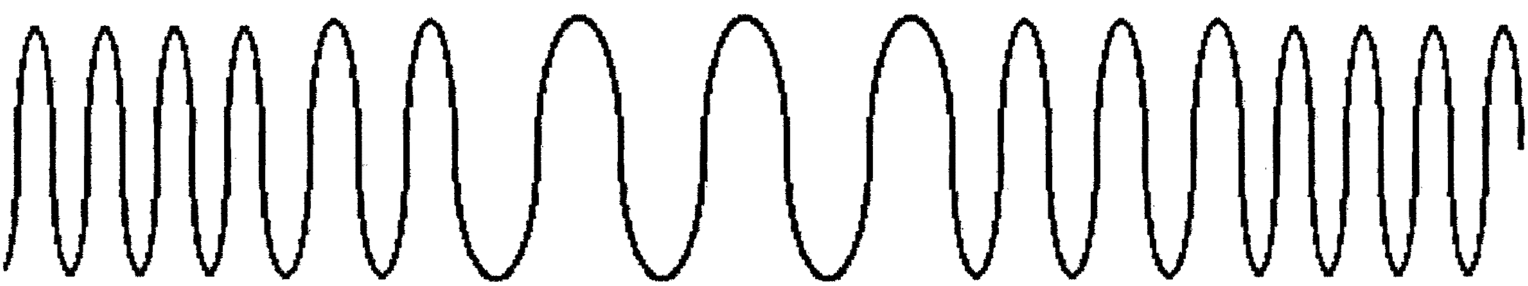
Separation of Scales in Waves

Real wave:





$$e^{i\frac{1}{\varepsilon}\theta(x)}$$



$$e^{i\frac{1}{\varepsilon}\theta(\varepsilon x)}$$