SYMMETRY, ITS BREAKING AND HIERARCHICAL LEVELS

Stephen Omohundro, CNLS
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ABSTRACT

Seen from the proper perspective, much of physics is a recognition of symmetry. The history of this perspective is briefly reviewed, leading to the modern geometrical formulation. Geometrical Hamiltonian mechanics, Noether's theorem and the general process of reduction to smaller spaces of description are reviewed. As examples, Euler's equations for the rigid body and the perfect fluid are shown to be identical. Breaking of symmetries often introduces a hierarchical structuring. A viewpoint is presented from which the guiding center and oscillation center descriptions in plasma physics are unified with geometrical optics descriptions in wave theories within a Hamiltonian framework. Connections with a general theory of different levels of description in physical theories are drawn, e.g., Kinetic vs. Fluid descriptions in statistical mechanics, quantum vs. classical physics, etc. Should time permit, relations with quantum chaos, topological excitations and catastrophe optics will be indicated.

Biographical Sketch

Stephen Omohundro is a graduate student in physics at U.C. Berkeley studying under Allan Kaufman in plasma physics and Alan Weinstein in mathematics. He is visiting CNLS for the summer.

Symmetry, Its Breaking, & Hierarchical Levels

by Stephen Omohundro
June 25, 1982, Center for Nonlinear Studies - Los Alamos, NM

1. Symmetry & Its History
2. Modern Hamiltonian Dynamics: Noether's Thm, Reduction
3. Lie-Poisson Brackets: Euler's eqns.
4. Hierarchical Structures & Symmetry Breaking
5. Averaging: Guiding & Oscillation center descriptions
6. Geometrical Optics: Diffraction, Quantum Chaos, Diffractals
7. General Theory of Levels of Description

PHILOSOPHY:

HOW IS PHYSICS POSSIBLE?

A priori you need at least one mark on the page for every thing in the real world, and any description is more complicated than reality.

SOLUTION: <u>SYMMETRY</u> !! - lots of things are the same.

Need symmetries of only parts of structures.

Can <u>define</u> a component of a structure this way.

eg. Approximate descriptions.

Like Klein in geometry: FOCUS ATTENTION ON THE <u>TRANSFORMATIONS</u> WHICH PRESERVE A STRUCTURE

$\Rightarrow$ CONCEPT OF A <u>GROUP</u>:

A set G with a multiplication $G \times G \to G$ such that:

1. $\exists$ Identity element e s.t. $ea = ae = a \quad \forall a \in G$
2. $\exists$ Inverses: $\forall a \in G$, $\exists a^{-1} \in G$ s.t. $aa^{-1} = a^{-1}a = e$
3. Associativity: $\forall a, b, c \in G \quad a(bc) = (ab)c$

HIGHLIGHTS IN GROUPS AND PHYSICS

<B.C. Interest in Symmetry - 5 Platonic Solids - 17 discrete symmetries of plan

1630 Galileo - relativity of motion

1770 Lagrange - develops some key concepts applied to number theory & solvability of eq

1815 Cauchy - permutation groups

1831 Galois - impossibility of quintic sln. - normal subgroup concept

1849 Cayley - formal definition of group - definition of matrices

1869 Sophus Lie - ideas of Galois applied to differential equations

1870 C. Jordan - "Traité's des substitutions et des equations algebriques" - 1st group th. book

1872 Felix Klein - Erlanger program for unifying geometry - publicity campaign for gr

1881 Weber - defines group representations

1894 Cartan - Classifies complex simple lie groups - real in 1914

~ Tait says of Cayley "Is it not a shame that such an outstanding man puts his abilities to such entirely useless questions?"

1904 Poincaré - notices symmetry group of special relativity

1918 Emmy Noether - theorem relating symmetries & conserved quantities

H. Weyl - Notices gauge symmetry of electromagnetism

1927 E. Wigner - uses representation theory to simplify perturbations in Q.M

1928 H. Weyl - publishes "Gruppentheorie und Quantenmechanick"

~ New theory jokingly referred to as "Group Pest"

1931 E. Wigner - publishes "Gruppentheorie"

1939 " - finds irreducible representations of inhomogeneous Lorentz gp

~1952 Applications to nuclear spectroscopy

1954 Yang & Mills - Non-abelian gauge symmetry

1962 Gell Mann & Zweig - SU(3) and quarks

1.

1962 Kirillov - 1-1 correspondence between co-adjoint orbits and irreducible unitary representations of simply connected nilpotent Lie groups

1964 Kostant - noticed co-adjoint orbits have symplectic structure

~1965 Books on special functions of physics in terms of groups

1966 Arnold - Rigid body & Perfect fluid as ham. systems on Lie groups

1968 Lax - Complete integrability of KdV & "Lax pair"

1970 Souriau - Momentum map & geometric quantization

1972 - Annual conference on "Group Theoretical Methods in Physics"

1974 Marsden & Weinstein - "Reduction" generalizing: symmetry ⇒ integrals

1979 - Adler - KdV on coadjoint orbits

1979 - Guillemin & Sternberg - Liquid drop model of nucleus as coadjoint orb.

1980 - Symes & Kostant - Integrability of Toda lattice as coadjoint orbi

1981 - Marsden & Weinstein - Maxwell-Vlasov eqns. of plasma as coadjoint orb

Notice the trend:

~1930 Symmetries of known physical systems simplify calculations
eg. Wigner-Eckart theorem

~1954 Symmetry is used to propose new physical model - Yang-Mills

~1962 Symmetry is basic attribute of model - SU(3) & quarks

~1968 Symmetries of fundamental structure not realized in nature
- spontaneous symmetry breaking - Higgs particles

~1970's Symmetries of kinematical structure independent of dynam

This is a realization of a viewpoint stressed by Weyl!

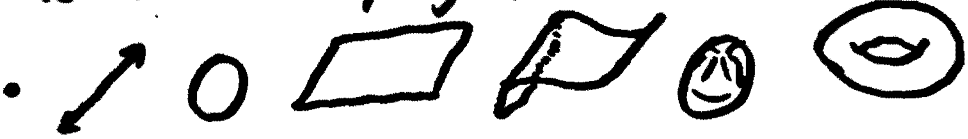
An **ACTION** OF A GROUP G ON A SET M is:
A map $G \times M \to M$ so that

1. $e \cdot m = m$ e = identity of G, $m \in M$
2. $g \cdot (h \cdot m) = (gh) \cdot m$ $\forall g, h \in G$, $m \in M$

eg. G = Rotations by $0°, 120°, 240°$ $M = \{$  $\}$

To be a symmetry, the action of G should preserve some structure of M

TAKE M TO BE THE STATE SPACE FOR SOME DYNAMICAL SYSTEM - typically endows M with a topological & differentiable structure

MANIFOLDS: 

LIE GROUPS eg. the real line under addition,
the torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$, $GL(n)$ = $n \times n$ nonsingular matrices

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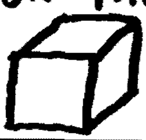
"Coordinates are unphysical" (but sometimes useful)
- they break the symmetry of a problem

eg.  vs. 

To understand a specific instance of something, look at a whole class of related things: eg. physical laws, orbit vs. phase portrait, perturbation theory, catastrophe theory, mathematics (eg. real numbers → topology & analysis, algebra, ordering.

MECHANICS:

Newton (1670): $F = ma$ for gravitating planets

Euler (1740): Universal principle (fluids, etc.)

Everything expressed in coordinates - only way to show configuration space coordinate independence was via a variational principle (Hamiltons principle)

Lagrange (1808): $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$

(Now known as Hamiltons equations)

Poisson (1830): $\{f, g\} = \sum_{i,j=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right)$

So $\dot{f} = \{f, H\}$

This structure is invariant under canonical transformations, generalizing contact transformations (eg. SHO 

)

MODERN FORMULATION - forget coordinates

A POISSON MANIFOLD is a manifold M with a

Poisson bracket: $\{\,,\}: \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$

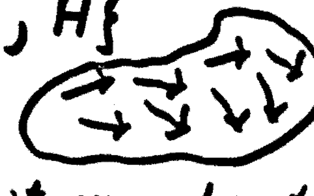
such that
1) bilinearity - $\{f, ag_1 + bg_2\} = a\{f, g_1\} + b\{f, g_2\}$, $a, b \in \mathbb{R}$, $f, g_1, g_2 \in \mathcal{F}$
2) antisymmetry - $\{f, g\} = -\{g, f\}$
3) Jacobi's identity - $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
4) Derivation property - $\{f, gh\} = \{f, g\}h + \{f, h\}g$

This structure is preserved under the (∞-dim) "Lie" group of canonical transformations.

<u>HAMILTON'S EQNS. OF MOTION</u>: $\dot{f} = \{f, H\}$

<u>HAMILTONIAN VECTOR FIELD</u>: $X_H = \{\cdot, H\}$

or in coordinates x^i on M: $X_H^i = \{x^i, H\}$



M

<u>A CONSERVED QUANTITY</u>: is a function that commutes w/ H

$\{f, H\} = 0$ so $\dot{f} = 0$, f const. on integral curves of X_H.

<u>A CASIMIR FUNCTION</u>: commutes with every function.

<u>A SYMPLECTIC MANIFOLD</u>: is a Poisson manifold where every tangent vector belongs to some hamiltonian vector field. (So the only Casimir functions are constants).

<u>THM</u> - Every Poisson manifold is foliated by symplectic leaves

A symmetry of a Poisson manifold M w/ hamiltonian H is the action of a Lie group G by canonical transformations on M which preserves the hamiltonian.

<u>eg</u> $G = \mathbb{R}^1$, the action is just the flow along a hamiltonian vector field X_f. H conserved $\Rightarrow X_f H = 0$

<u>Noether's Theorem</u>: $\Rightarrow f$ is a conserved quantity. <u>Pf</u>: $\dot{f} = \{f, H\} = -\{H, f\} = -X_f H = 0$

Generalization: <u>REDUCTION</u> (Marsden & Weinstein)

Orbit space M/G inherits a poisson structure, allows one to reduce the number of degrees of freedom under consideration.

eg if G is commutative:  → translation symmetry

Use Noether: p_x const., x-irrelevant lose 2-degrees of freedom

commutative ⇒ still y, z translation symmetric ⇒ get rid of 6 deg.

eg if G not commutative - more complicated

same as above, rotation symmetry. L_x, L_y, L_z conserved

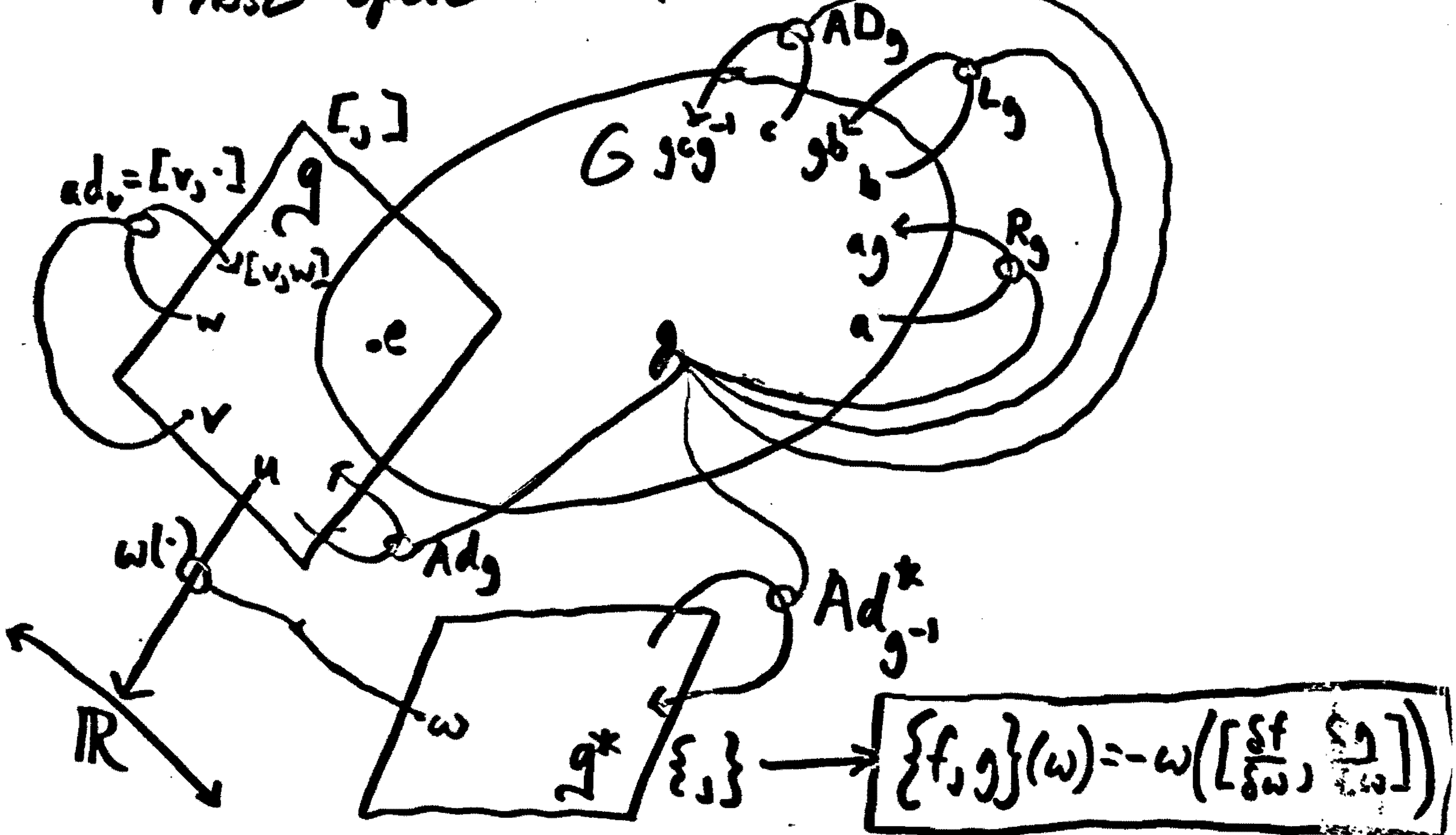
so fix them but once we mod out by, say rotations about z

then have picked a direction - can't mod out by the others

⇒ only get rid of 4 degrees of freedom

PUN - Let G be the configuration space for a system (as well as the symmetry group.

Phase space = T^*G

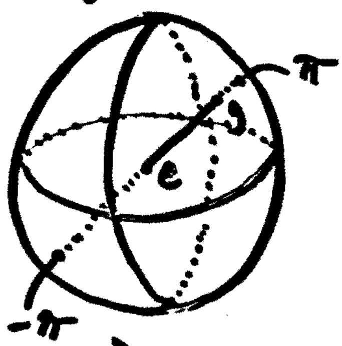


$$\boxed{\{f, g\}(\omega) = -\omega\left(\left[\frac{\delta f}{\delta \omega}, \frac{\delta g}{\delta \omega}\right]\right)}$$

Get Lie-Poisson bracket on $\mathfrak{g}^*$:

x. <u>THE RIGID BODY</u> – Pick a starting position, the configuration space is then the rotation group $SO(3)$

 <u>Topologically</u> $= \mathbb{RP}^3$



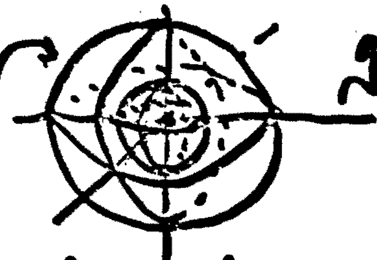
(Topology important since not simply connected i.e.  ⇒ double-valued spin repns. ⇒ Pauli exclusion principle ⇒ US!)

<u>Lie algebra</u> $= \mathfrak{so}(3) =$ 3×3 anti-symmetric matrices

Identify with $\mathbb{R}^3$ via $u = \begin{pmatrix} 0 & -\vec{u}_z & \vec{u}_y \\ \vec{u}_z & 0 & -\vec{u}_x \\ -\vec{u}_y & \vec{u}_x & 0 \end{pmatrix}$ $\quad [u,v] = \vec{u} \times \vec{v}$

<u>Bi-invariant metric</u> $\vec{u} \cdot \vec{v} = -\frac{1}{2}\,\mathrm{Trace}(uv)$

<u>Co-adjoint orbits</u> = symplectic leaves



$\mathfrak{g}^* \simeq \mathbb{R}^3$

$\mathfrak{g}^*$ is the space of angular momenta in the body. The coadjoint orbits are spheres of constant angular momentum.

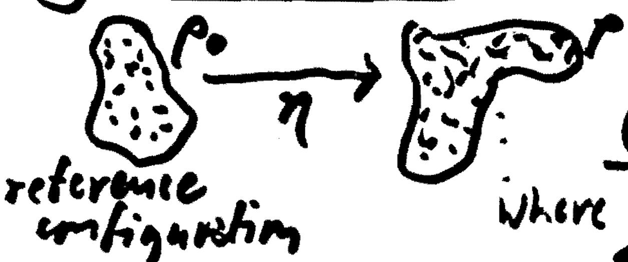
⇒ total angular momentum is a Casimir function which will be conserved by <u>any</u> hamiltonian flow on $\mathfrak{g}^*$. In coordinates $(L_1, L_2, L_3) \in \mathfrak{g}^*$ with the usual hamiltonian: $H(\vec{L}) = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right)$

The Lie-Poisson equations become the Euler eqns:

$$\dot{\vec{L}} = \{\vec{L}, H\} = -L\left(\left[\frac{\delta \vec{L}}{\delta L}, \frac{\delta H}{\delta L}\right]\right) = -\vec{L} \cdot \left(\vec{L} \times \frac{\delta H}{\delta L}\right)$$

or $\quad \dot{L}_1 = \frac{I_2 - I_3}{I_2 I_3} L_2 L_3$ + cyclic permutations

eg COMPRESSIBLE IDEAL FLUID



Group: $G = \mathcal{D} \odot \mathcal{F}$

where $\mathcal{D}$ = diffeomorphisms of $\mathbb{R}^3$

$\mathcal{F}$ = functions on $\mathbb{R}^3$

$\odot$ = semi-direct product, w/ usual action of $\mathcal{D}$ on $\mathcal{F}$

Lie Algebra: $\mathfrak{g} = \mathfrak{X} \times \mathcal{F}$ $\qquad \mathfrak{X}$ = vector fields on $\mathbb{R}^3$

Dual of Lie Algebra: $\mathfrak{g}^* = \mathfrak{X}^* \times \mathcal{F}^*$ $\qquad \mathfrak{X}^*$ = momentum densities

$\mathcal{F}^*$ = mass densities

Lie Poisson product: $F, G : \mathfrak{X}^* \times \mathcal{F}^* \to \mathbb{R}$, $m \in \mathfrak{X}^*$, $\rho \in \mathcal{F}^*$

$$\{F,G\}(m,\rho) = -\int (m,\rho)\left[\left(\frac{\delta F}{\delta m}, \frac{\delta F}{\delta \rho}\right), \left(\frac{\delta G}{\delta m}, \frac{\delta G}{\delta \rho}\right)\right] d^3x$$

$$= -\int (m,\rho)\left(\left[\frac{\delta F}{\delta m}, \frac{\delta G}{\delta m}\right], \frac{\delta F}{\delta m}\left(\frac{\delta G}{\delta \rho}\right) - \frac{\delta G}{\delta m}\left(\frac{\delta F}{\delta \rho}\right)\right) d^3x$$

$$= -\int m \cdot \left[\frac{\delta F}{\delta m}, \frac{\delta G}{\delta m}\right] d^3x - \int \rho \left(\left(\nabla \frac{\delta G}{\delta \rho}\right) \cdot \frac{\delta F}{\delta m} - \frac{\delta G}{\delta m} \cdot \nabla \frac{\delta F}{\delta \rho}\right) d^3x$$

$$= -\int \left(m \left(\frac{\delta F}{\delta m} \cdot \nabla \frac{\delta G}{\delta m} - \frac{\delta G}{\delta m} \cdot \nabla \frac{\delta F}{\delta m}\right) + \rho \left(\frac{\delta F}{\delta m} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta m} \cdot \nabla \frac{\delta F}{\delta \rho}\right)\right) d^3x$$

Hamiltonian: $H : \mathfrak{X}^* \times \mathcal{F}^* \to \mathbb{R}$ $\qquad H(v,\rho) = \frac{1}{2}\int (\rho v^2 + \rho U(\rho))\, d^3x$

Eqns. of Motion: $\dot{F} = \{F, H\}$ = Euler's Eqns!

$$\frac{\partial v}{\partial t} + v \cdot \nabla v + \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial U}{\partial \rho}\right) = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0$$

This is analogous to a top in a gravitational field.

Easy to add in entropy and any other convecting quantities by enlarging the group.

FURTHER EXAMPLES (from Marsden 1981)

System	Group
1. Free rigid body	$SO(3)$ = rotation group
2. heavy top	$E(3)$ = Euclidean group
3. perfect incompressible fluid	$\mathcal{D}_{vol}$ = volume preserving diffeomorphisms
4. compressible fluid	$\mathcal{D} \ltimes \mathcal{F}$ = diffeomorphisms semi-direct x functions
5. Korteweg de Vries equation	$\mathcal{A}\mathcal{F}$ = invertible Fourier Integral operators
6. Toda Lattice	lower triangular matrices
7. Liouville equation	$\mathcal{S}$ = canonical transformations
8. Heisenberg equations of Q.M.	$\mathcal{U}(\mathcal{H})$ = unitary transformations of $\mathcal{H}$
9. Lax eqns. of nonlinear waves	$\mathcal{U}$ = unitary group
10. Poisson-Vlasov equations	$\mathcal{S}$ = canonical transformations

Other examples arise through symmetry & reduction

11. Maxwell's Equations	gauge group of electrodynamics
12. Yang-Mills' Equations	principle bundle automorphisms
13. Einstein's eqn. in general relativity	diffeomorphism group of spacetime
14. Supergravity	supersymmetry transformations

THE UNIVERSE IS FULL OF HIERARCHICAL STRUCTURE

eg matter: quantum fields → particle excitations (eg quarks) → composite particles (eg. hadrons) → atoms → molecules → crystal grains → boulders → planets & stars → galaxies → clusters → superclusters → ?

eg levels of description in physics

eg knowledge physics → chemistry → biology → psychology → sociology

eg speech tones → phonemes → words → sentences → stories → speeches

eg biology organelles → cells → tissues → organs → organisms → societies → eco-systems

eg political systems, machines, levels of command etc. etc. etc.

In physics (and many of the other systems), the basic simplification of hierarchies is:

BEHAVIORS ON SCALES LARGER THAN YOURS ARE VARYING SO SLOWLY IN SPACE AND TIME THAT THEY AFFECT YOU ALMOST AS IF THEY WERE CONSTANT.

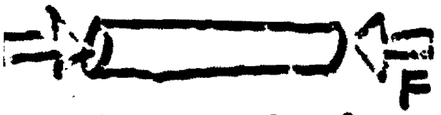
BEHAVIORS ON SCALES SMALLER THAN YOURS ARE VARYING SO QUICKLY IN SPACE AND TIME THAT ALMOST ONLY THEIR AVERAGE EFFECT IS IMPORTANT.

Such an idea is responsible for most of the perturbation theories in use – one would like to find a general setting in which to make it precise

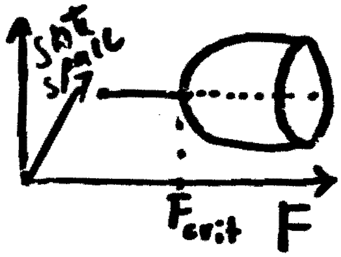
IN A HAMILTONIAN SYSTEM, APPROXIMATE SYMMETRIES GIVE RISE TO APPROXIMATELY CONSERVED QUANTITIES.

$|X_f \cdot H| = \varepsilon \Rightarrow |\dot{f}| = |\{f, H\}| = |\{H, f\}| = |X_f \cdot H| = \varepsilon$

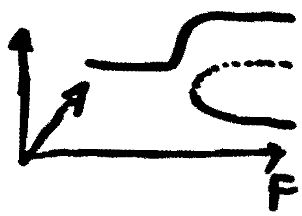
If one uses bifurcation theory to study the statics of a compressed cylindrically symmetric beam



one obtains a bifurcation diagram where a stable fixed point gives birth to an unstable fixed point and a whole circle of stable (marginally in some directions) fixed points:



If one breaks the symmetry one obtains a diagram:



Really doing the experiment one finds that for small assymmetry the beam bends quickly in any of the original directions, but then slowly drifts to the preferred direction.

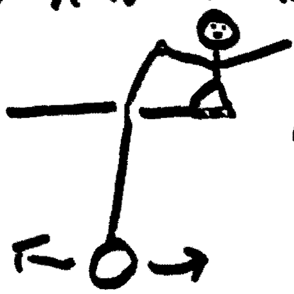
I.e. the symmetry breaking has introduced <u>dynamics</u> on a new time scale relative to the original problem.

This is a very general phenomenon.

Theorem (modulo some technicalities): Given a dynamical system X_ε $0 \le \varepsilon$ whose flow reduces to a non-zero circle action at $\varepsilon = 0$, one can find an action of S^1 dependent on ε which preserves X_ε for some interval of ε.

This carries over to the hamiltonian case where the S^1 action is generated by a conserved quantity J_ε.

The first order piece of J_ε in an ε Taylor expansion is usually known as the adiabatic invariant.

Example  pendulum w/ slowly varying length

$\frac{\text{energy}}{\text{freq.}} = \frac{H}{\omega}$ is adiabatically conserved.

Ex Gyro-motion of charged particle in a magnetic field.

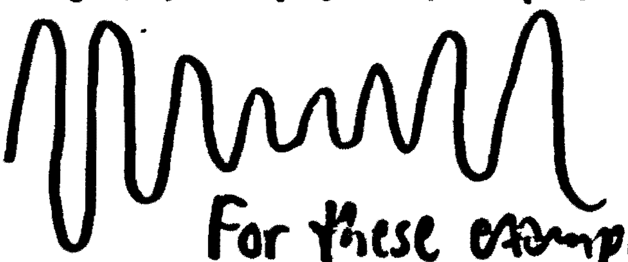
e.g. 2-d (x, y, P_x, P_y) $\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial P_x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial P_y} - \frac{\partial f}{\partial P_x}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial P_y}\frac{\partial g}{\partial y}$

$\underline{B} = \nabla \times \underline{A}$ $\qquad H = \frac{1}{2}(\underline{p} - \frac{e}{c}\underline{A})^2$

Or: $(\underline{x}, \underline{v})$ $\quad \{f, g\} = \frac{\partial f}{\partial \underline{x}} \cdot \frac{\partial g}{\partial \underline{v}} - \frac{\partial f}{\partial \underline{v}}\frac{\partial g}{\partial \underline{x}} + \underline{B} \cdot \left(\frac{\partial f}{\partial \underline{v}} \times \frac{\partial g}{\partial \underline{v}}\right)$

$H = \frac{1}{2}v^2$

Ex. Oscillation Center motion of charged particle in E&M wave



Ponderomotive Potential

For these examples "Lie Transform" methods

Consider linear waves in a homogeneous medium. The translation group symmetry lifts (analogous to Koopmanism) to a linear representation on wave phase space. This action decomposes into an ∞-dim torus action (fourier components just get phase shifted) thus the single freq. ω subspace has a circle action on it.

$\Rightarrow$ ripe for perturbing.

Slowly varying speed in medium $-\nabla^2 u + n(\underline{x}) u_{tt} = 0$

Still hamiltonian: $u_t = \frac{w}{n(\underline{x})}$ $\quad w_t = \nabla^2 u$

$$\{\mathcal{F}, \mathcal{G}\} = \int \left(\frac{\delta \mathcal{F}}{\delta u} \frac{\delta \mathcal{G}}{\delta w} - \frac{\delta \mathcal{F}}{\delta w} \frac{\delta \mathcal{G}}{\delta u} \right) d^3x \qquad H = \int \left(\frac{1}{2} \frac{w^2}{n(\underline{x})} + \frac{1}{2} \nabla u \cdot \nabla u \right) d^3x$$

$\Rightarrow$ Geometric Optics, WKB, etc. $\quad u = \tilde{u}\, e^{i\psi(x)}$

eikonal eqn. for rays from ψ.

transport eqns. for amplitude $\tilde{u}$.

Write waves on k, x space using local fourier transform (Wigner functions): $u^2(k,x) = \int e^{i\underline{k}\cdot\underline{x}'} u(\underline{x} + \frac{1}{2}\underline{x}') u^*(\underline{x} - \frac{1}{2}\underline{x}')\, d^3x'$

Obtain evolution in terms of a Lie-Poisson bracket.

$$\{A_1, A_2\} = \int d^3k\, d^3x\, \mathcal{D}(\underline{k}, \underline{x}) \left[\frac{\partial}{\partial \underline{x}} \left(\frac{\delta A_1}{\delta \mathcal{D}} \right) \cdot \frac{\partial}{\partial \underline{k}} \left(\frac{\delta A_2}{\delta \mathcal{D}} \right) - \frac{\partial}{\partial \underline{k}} \left(\frac{\delta A_1}{\delta \mathcal{D}} \right) \cdot \frac{\partial}{\partial \underline{x}} \left(\frac{\delta A_2}{\delta \mathcal{D}} \right) \right]$$

Group is symplectomorphisms of $\underline{k}, \underline{x}$ space. A coadjoint orbit is densities w/ support on Lagrangian submanifolds i.e. via Maslov eikonal waves (which have a canonical bracket

Some Examples of Levels of Description in Physics

1. Quantum Electrodynamics → Quantum particles in classical E&M → exact classical particles in E&M → non-relativistic particles in E&M → macroscopic matter (D, H) + E&M → eikonal waves = geometrical optics → optical system as symplectic transformation (eg. of θ's + positio

2. Quantum (e^- + nucleus) → Quantum e^- + classical nucleus → classical (e^- + nucleus

3. Particle desc. of fluid → viscous fluid → inviscid fluid → shallow waves → KdV →

4. Klimontovich plasma → BBGKY hierarchy → Vlasov → Fluid, → Eikonal, → Osc. Center etc.

5. 3-body problem → restricted 3-body problem → 2-body problem

6. E&M → electric circuits

7. General Relativity → Special Relativity → Newtonian dyn. + Galilean re

8. BCS superconductivity → London picture

9. Statistical Mechanics → Thermodynamics (Non-eqn. → eqn.)

10. Crystals → phonons + dislocations (Kosterlitz-Thouless phase tra

11. Heisenberg spin system → "vortices" + spin waves

12. Gas Dynamics → Nonlinear acoustics → linear acoustics

13. Nucleus → Liquid Drop model

14. 2-d fluid → vortices

15. particulate description → rigid bodies (classical classical mech

- particles + field → pseudoparticles (many-body theory)